

# Components of the Hilbert scheme of space curves on low-degree smooth surfaces

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With an appendix by John C. Ottem

## Abstract

We study maximal families  $W$  of the Hilbert scheme,  $H(d, g)_{sc}$ , of smooth connected space curves whose general curve  $C$  lies on a smooth surface  $S$  of degree  $s$ . For every integer  $s$  we give conditions on  $C$  under which  $W$  is a generically smooth component of  $H(d, g)_{sc}$  and we determine  $\dim W$ . If  $s = 4$  and  $W$  is an irreducible component of  $H(d, g)_{sc}$ , then the Picard number of  $S$  is at most 2 and we explicitly describe non-reduced and generically smooth components in the case  $\text{Pic}(S)$  is generated by the classes of a line and a smooth plane cubic curve. For curves on smooth cubic surfaces we find new classes of non-reduced components of  $H(d, g)_{sc}$ , thus making progress in proving a conjecture for such families.

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## 1 Introduction and Main Results

In this paper we study the Hilbert scheme of smooth connected space curves,  $H(d, g)_{sc}$ , with regard to dimension and smoothness, with a special look to existence of non-reduced components. There are several papers that consider such problems, see e.g. [1], [3, 4, 6], [13, 14], [17–22], [25–27, 29, 30] and the book [15]. Here we generalize the approach that was used in [17] for curves on cubic surfaces to study families of curves on smooth surfaces of degree  $s$ . In particular we investigate when maximal subsets  $W$  of the Hilbert scheme  $H(d, g)_{sc}$  whose general curve  $C$  lies on a quartic surface, form non-reduced, or generically smooth, irreducible components of  $H(d, g)_{sc}$ . We find a pattern similar to what is known for maximal irreducible families of curves on smooth cubic surfaces; if  $H^1(\mathcal{I}_C(s)) = 0$ , then  $W$  turns out to be a generically smooth component of  $H(d, g)_{sc}$ . If, however,  $H^1(\mathcal{I}_C(s)) \neq 0$  and the genus is sufficiently large, then  $W$  is still an irreducible component, but it is now non-reduced. For  $s = 4$  it suffices to take “ $g$  large” as  $g > G(d, 5)$ , the maximum genus of curves of degree  $d$  not contained in a degree-4 surface (see (13)), or as the better bound

$$g > \min\{G(d, 5) - 1, \frac{d^2}{10} + 21\} \quad \text{and} \quad d \geq 21, \quad (1)$$

see Theorem 3.1 and Corollary 3.5 of Section 3.

If  $W$  is a closed subset of  $H(d, g)_{sc}$ , we denote by  $s(W)$  the minimal degree of a surface containing a general curve of  $W$ . As in [17] we say  $W$  is  $s(W)$ -maximal if it is irreducible and maximal with respect to  $s(W)$ , i.e.  $s(V) > s(W)$  for any closed irreducible subset  $V$  properly containing  $W$ . We say  $W$  is an  $s(W)$ -maximal family or subset of  $H(d, g)_{sc}$  in this case. By Remark 2.5 below, if a very general curve of a 4-maximal family  $W$  (e.g. an irreducible component of  $H(d, g)_{sc}$ ) sits on a smooth quartic surface  $S$  and  $d > 16$ , then the Picard number of  $S$  is *at most* 2.

A case we consider closely is a smooth quartic surface  $S \subset \mathbb{P}^3$  where the Picard group  $\text{Pic}(S)$  is generated over  $\mathbb{Z}$  by the classes of two smooth connected curves  $\Gamma_1$  and  $\Gamma_2$  satisfying  $\Gamma_1^2 = -2$ ,  $\Gamma_2^2 = 0$ ,  $\Gamma_1 \cdot \Gamma_2 = 3$ , i.e. with intersection matrix  $\begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix}$ , and such that  $H = \Gamma_1 + \Gamma_2$  is a hyperplane

section. The existence of surfaces with the properties that we need to show the theorem below, is proved by Ottem in the appendix. Writing the divisor class of  $a\Gamma_1 + b\Gamma_2$  as  $(a, b)$  and a curve  $C$  (i.e. effective divisor) in its linear system shortly as  $C \equiv (a, b)$ , then  $a, b \geq 0$ . In Section 4 we prove:

**Theorem 1.1.** *Let  $S \subset \mathbb{P}^3$  be a smooth quartic surface with  $\Gamma_1, \Gamma_2$  as above, let  $C \equiv (a, b)$  be a smooth connected curve and suppose  $a \neq b$  and  $d > 16$ . Then  $C$  belongs to a unique 4-maximal family  $W \subseteq H(d, g)_{sc}$ . Moreover if  $\tilde{S}$  is a quartic surface containing a very general member of  $W$ , then  $\text{Pic}(\tilde{S})$  is freely generated by the classes of a line and a smooth plane cubic curve, and every  $C \equiv (a, b)$  contained in some surface  $S$  as above belongs to  $W$ . Furthermore  $\dim W = g + 33$ ,*

$$d = a + 3b, \quad g = 3ab - a^2 + 1 \quad \text{and}$$

I)  $W$  is a generically smooth, irreducible component of  $H(d, g)_{sc}$  provided  $4 < a < \frac{3b}{2} - 1$ .

II)  $W$  is a non-reduced irreducible component of  $H(d, g)_{sc}$  provided

$$\frac{3b}{2} - 1 \leq a \leq \frac{3b}{2} \tag{2}$$

and (1) holds. Explicitly, this region is given by the three families

$$a) (8 + 3k, 6 + 2k) \quad b) (10 + 3k, 7 + 2k) \quad c) (15 + 3k, 10 + 2k) \text{ for } k \geq 0.$$

One may show that  $W$  is non-empty, i.e. that there exist smooth connected curves  $C \equiv (a, b)$  if and only if  $0 < a \leq \frac{3b}{2}$ , or  $(a, b) \in \{(1, 0), (0, 1)\}$ . The case  $a = b$  corresponds to  $C$  being a complete intersection of  $S$  with some other surface (a c.i. in  $S$ ). Moreover if  $C$  belongs to the family a) (resp. b), c)) in Theorem 1.1, we have  $h^1(I_C(4)) = 1$  (resp.  $h^1(I_C(4)) = 2$ ,  $h^1(I_C(4)) = 4$ ) where the numbers  $\dim W + h^1(I_C(4))$  are precisely the dimension of the tangent space of  $H(d, g)_{sc}$  at  $(C)$  by (7)!

Another main result of this paper (Theorem 2.2) is related to I) above. Specializing to surfaces of degree  $s \leq 4$ , we get:

**Proposition 1.2.** *Let  $s$  be an integer,  $1 \leq s \leq 4$  and let  $W \subseteq H(d, g)_{sc}$  be an  $s$ -maximal family such that  $d > s^2$ . Let  $C$  be a member of  $W$  sitting on a smooth surface  $S$  of degree  $s$  satisfying  $H^1(\mathcal{I}_C(s)) = 0$ . If  $s = 4$  suppose also that  $C$  is not a c.i. in  $S$ . Then  $W$  is a generically smooth irreducible component of  $H(d, g)_{sc}$  of dimension  $(4 - s)d + g + \binom{s+3}{3} - 2$ .*

In Theorem 2.2 we get almost the same conclusion for every integer  $s \geq 1$  under the assumption

$$H^1(\mathcal{I}_C(s)) = 0 \quad \text{and} \quad H^1(\mathcal{I}_C(s - 4)) = 0$$

provided we in addition, as in Theorem 1.1, suppose that  $C \equiv eE + fH$ ,  $e, f \in \mathbb{Z}$  where  $E$  is e.g. an arithmetically Cohen-Macaulay (ACM) curve.

Finally we consider in Section 5 a conjecture about non-reduced components for maximal families  $W \subset H(d, g)_{sc}$  of linearly normal curves on a smooth cubic surface  $S$  [17, Conj. 4]. The conjecture states that if the 7-tuple of the invertible sheaf  $\mathcal{O}_S(C)$  of  $\text{Pic}(S)$  of a general member  $C$  of  $W$  satisfies  $\delta \geq m_1 \geq \dots \geq m_6$  and  $\delta \geq m_1 + m_2 + m_3$  (cf. that section for the notation), then  $W$  is a non-reduced irreducible component if and only if

$$d \geq 14, \quad 3d - 18 \leq g \leq (d^2 - 4)/8 \quad \text{and} \quad 1 \leq m_6 \leq 2,$$

where the inequality for  $m_6$  is equivalent to  $H^1(\mathcal{I}_C(3)) \neq 0$ . In Theorem 5.3 we prove the conjecture for  $m_5 \geq 6 - m_6$  with a few possible exceptions. This result was mainly lectured at a workshop at

the Emile Borel Center, Paris in May 1995, and may be known to some experts in the field, but it has not been published. We thank O. A. Laudal for interesting discussions on that subject.

To get Theorem 1.1 the contribution in the appendix by Ottem is significant because he there explicitly describes the quartic surface of the theorem with the desired properties, as well as determining the range where  $H^1(\mathcal{I}_C(4))$  is trivial or non-trivial. We thank J.C. Ottem for writing this appendix on quartic surfaces, and for his remarks to other aspects of this paper too. We also thank D. Eklund for our discussion on the Picard number of smooth K3 surfaces and R. Hartshorne for his comments.

## 1.1 Notations and terminology

In this paper the ground field  $k$  is supposed to be *algebraically closed of characteristic zero* (and uncountable in the statements where the concept "very general" is used). A surface  $S$  in  $\mathbb{P}^3$  is a hypersurface, defined by a single equation. A curve  $C$  in  $\mathbb{P}^3$  (resp. in  $S$ ) is a *pure one-dimensional* subscheme of  $\mathbb{P} := \mathbb{P}^3$  (resp.  $S$ ) with ideal sheaf  $\mathcal{I}_C$  (resp.  $\mathcal{I}_{C/S}$ ) and normal sheaf  $\mathcal{N}_C = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{O}_C)$  (resp.  $\mathcal{N}_{C/S} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_{C/S}, \mathcal{O}_C)$ ). We denote by  $d = d(C)$  (resp.  $g = g(C)$ ) the degree (resp. arithmetic genus) of  $C$ . If  $\mathcal{F}$  is a coherent  $\mathcal{O}_{\mathbb{P}}$ -Module, we let  $H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$ ,  $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$ ,  $\chi(\mathcal{F}) = \sum (-1)^i h^i(\mathcal{F})$  and we often write  $H^i(S, \mathcal{O}_S(C))$  as  $H^i(\mathcal{O}_S(C))$  for a Cartier divisor  $C$  on  $S$ . Let

$$s(C) = \min\{n \mid h^0(\mathcal{I}_C(n)) \neq 0\}.$$

We denote by  $H(d, g)$  (resp.  $H(d, g)_{sc}$ ) the Hilbert scheme of (resp. smooth connected) space curves of Hilbert polynomial  $\chi(\mathcal{O}_C(t)) = dt + 1 - g$  [7]. The curve in a small enough open irreducible subset  $U$  of  $H(d, g)$  is called a *general* curve of  $H(d, g)$ . So any member of  $U$  has all the openness properties which we want to require. A *generization*  $C' \subseteq \mathbb{P}^3$  of  $C \subseteq \mathbb{P}^3$  in  $H(d, g)$  is the general curve of some irreducible subset of  $H(d, g)$  containing  $(C)$ . By an irreducible component of  $H(d, g)$  we always mean a *non-embedded* irreducible component. We denote by  $H(s)$  the Hilbert scheme of surfaces of degree  $s$  in  $\mathbb{P}^3$ . A member of a closed irreducible subset  $V$  of  $H(s)$  or  $H(d, g)_{sc}$  is called *very general* in  $V$  if it is smooth and sits outside a countable union of proper closed subset of  $V$ .

## 2 Background

In this section we first recall some results from [17] needed in this paper. Their proofs uses the deformation theory developed by Laudal in [23]; in particular the results rely on [23, Thm. 4.1.4].

Let  $D(d, g; s)$  (resp.  $D(d, g; s)_{sc}$ ) be the Hilbert-flag scheme parameterizing pairs  $(C, S)$  of curves (resp. smooth connected curves)  $C$  contained in surfaces  $S$  in  $\mathbb{P}^3$  with Hilbert polynomials  $p(t) = dt + 1 - g$  and  $q(t) = \binom{t+3}{3} - \binom{t-s+3}{3}$  respectively. Then the tangent space,  $A^1 := A^1(C \subset S)$ , of  $D(d, g; s)$  at  $(C, S)$  is given by the Cartesian diagram

$$\begin{array}{ccccccc} A^1 & \longrightarrow & H^0(\mathcal{N}_S) & \simeq & H^0(\mathcal{O}_S(s)) \\ & & \downarrow & \square & \downarrow m \\ 0 \rightarrow H^0(\mathcal{N}_{C/S}) & \rightarrow & H^0(\mathcal{N}_C) & \rightarrow & H^0(\mathcal{N}_S|_C) & \simeq & H^0(\mathcal{O}_C(s)) \end{array} \quad (3)$$

where the morphisms are induced by natural (or restriction) maps to normal sheaves.

Suppose  $S$  is a smooth surface of degree  $s$ . If  $C$  is a curve (Cartier divisor), we have  $\mathcal{N}_{C/S} \simeq \omega_C \otimes \omega_S^{-1}$  and a connecting homomorphism  $\delta : H^0(\mathcal{N}_S|_C) \rightarrow H^1(\mathcal{N}_{C/S}) \simeq H^0(\mathcal{O}_C(s-4))^\vee$  continuing the lower horizontal sequence in (3). Let  $\alpha = \alpha_C := \delta \circ m$  be the composed map and let  $A^2 := \text{coker } \alpha$ . Using (3) we easily get  $\dim A^1 - \dim A^2 = (4-s)d + g + \binom{s+3}{3} - 2$  and an exact sequence

$$0 \rightarrow H^0(\mathcal{I}_{C/S}(s)) \rightarrow A^1 \rightarrow H^0(\mathcal{N}_C) \rightarrow H^1(\mathcal{I}_C(s)) \rightarrow \text{coker } \alpha_C \rightarrow H^1(\mathcal{N}_C) \rightarrow H^1(\mathcal{O}_C(s)) \rightarrow 0. \quad (4)$$

The map  $A^1 \rightarrow H^0(\mathcal{N}_C)$  in (3) and (4) is the tangent map of the 1<sup>st</sup> projection,

$$pr_1 : D(d, g; s) \longrightarrow H(d, g), \quad \text{induced by } pr_1((C_1, S_1)) = (C_1), \quad (5)$$

at  $(C, S)$ . Since we may look upon  $D(d, g; s)$  as a relative Hilbert scheme over  $H(d, g)$  (cf. [15, Thm. 24.7]), it follows that  $pr_1$  is a projective morphism by [7]. By [17, Lem. A10]  $pr_1$  is smooth at  $(C, S)$  under the assumption

$$H^1(\mathcal{I}_C(s)) = 0. \quad (6)$$

Moreover by [17, (2.6)]  $A^2 = \text{coker } \alpha_C$  contains the obstructions of deforming the pair  $(C, S)$ .

Let  $C$  be a smooth connected curve. If we suppose  $d > s^2$ , then it is easy to see  $H^0(\mathcal{I}_{C/S}(s)) = 0$  and that *the restricted* projection,  $pr_1 : D(d, g; s)_{sc} \rightarrow H(d, g)_{sc}$ , is injective in  $pr_1^{-1}(U)$  for some neighborhood  $U \subset H(d, g)_{sc}$  of  $(C)$ . An  $s(W)$ -maximal (or just maximal) family  $W$  of  $H(d, g)_{sc}$  containing  $(C)$  is therefore nothing but the image under  $pr_1$  of an irreducible component of  $D(d, g; s)_{sc}$  containing  $(C, S)$  ([19, Def. 1.24 and Cor. 1.26]). If we in addition suppose that  $\alpha$  is surjective, then  $(C, S)$  belongs to a unique generically smooth component of  $D(d, g; s)_{sc}$  and

$$\dim W = h^0(\mathcal{N}_C) - h^1(\mathcal{I}_C(s)) = (4 - s)d + g + \binom{s+3}{3} - 2. \quad (7)$$

Assuming also (6) it follows that  $W$  is a generically smooth irreducible component of  $H(d, g)_{sc}$  ([17, Thm. 10]).

Using the infinitesimal Noether-Lefschetz theorem for  $s = 4$  ([11, p. 253]), we get that  $\alpha$  is surjective provided  $s \leq 4$ ,  $d > s^2$  and  $C$  is smooth and connected, but (for  $s = 4$  only) not a complete intersection of  $S$  with some other surface (i.e., not a c.i. in  $S$ , or equivalently  $2g - 2 \neq d(s - 4 + d/s)$ ), cf. [17, Lem. 13 and Thm. 1]. Hence  $D(d, g; s)$  is smooth at  $(C, S)$  and we get all conclusions above, assuming (6) for the final one. As a consequence we obtain Proposition 1.2.

**Remark 2.1.** *Let  $S$  be smooth,  $d > s^2$  and  $s \leq 4$ . If  $W$  is an irreducible component of  $H(d, g)_{sc}$  containing  $(C)$ , then  $\dim W = (4 - s)d + g + \binom{s+3}{3} - 2 \geq 4d$ , i.e.*

$$g \geq sd - \binom{s+3}{3} + 2, \quad (8)$$

or equivalently,  $h^1(\mathcal{I}_C(s)) \leq h^1(\mathcal{O}_C(s))$ . Moreover if the general curve of  $W$  does not satisfy (6), we get by (7) that the component  $W$  is non-reduced (i.e. not generically smooth) and that (8) holds.

We also need to consider the Hilbert scheme,  $H(s)$ , of surfaces of degree  $s$  in  $\mathbb{P}^3$  and the 2<sup>nd</sup> projection;

$$pr_2 : D(d, g; s) \longrightarrow H(s), \quad \text{induced by } pr_2((C_1, S_1)) = (S_1).$$

Moreover let  $\text{Pic}$  be the relative Picard scheme over the open set in  $H(s)$  of smooth surfaces of degree  $s$ , see [8]. Then there is a projection  $p_2 : \text{Pic} \rightarrow H(s)$ , forgetting the invertible sheaf, and a rational map,

$$\pi : D(d, g; s) \dashrightarrow \text{Pic}, \quad \text{induced by } \pi((C_1, S_1)) = (\mathcal{O}_{S_1}(C_1), S_1) \quad (9)$$

which is defined on the open subscheme  $U \subset D(d, g; s)$  given by pairs  $(C_1, S_1)$  where  $C_1$  is Cartier on a smooth  $S_1$ . Obviously, if we restrict to  $U$  we have  $p_2 \circ \pi = pr_2$ . Note that if  $H^1(S, \mathcal{O}_S(C)) \simeq H^1(\mathbb{P}^3, \mathcal{I}_C(s - 4))^\vee = 0$  then  $\pi$  is smooth at  $(C, S)$  by [8, Rem. 4.5]. Indeed  $H^1(S, \mathcal{L}) = 0$ ,  $\mathcal{L} := \mathcal{O}_S(C)$ , implies a surjective map  $A^1 \rightarrow T_{\text{Pic}, \mathcal{L}}$  between the tangent spaces of  $D(d, g; s)$  at

$(C, S)$  and  $\text{Pic}$  at  $(\mathcal{L})$  and an injection  $\text{coker } \alpha_C \rightarrow \text{coker } \alpha_{\mathcal{L}}$  on their obstruction spaces (mapping obstructions onto obstructions), fitting into the following diagram of exact horizontal sequences

$$\begin{array}{ccccccc} 0 \rightarrow H^0(\mathcal{N}_{C/S}) & \longrightarrow & A^1 & \longrightarrow & H^0(\mathcal{N}_S) & \xrightarrow{\alpha_C} & H^1(\mathcal{N}_{C/S}) \\ & \downarrow & \circ & \downarrow & \parallel & \circ & \downarrow \\ 0 \rightarrow H^1(\mathcal{O}_S) = 0 & \longrightarrow & T_{\text{Pic}, \mathcal{L}} & \longrightarrow & H^0(\mathcal{N}_S) & \xrightarrow{\alpha_{\mathcal{L}}} & H^2(\mathcal{O}_S) . \end{array} \quad (10)$$

Here  $\alpha_{\mathcal{L}}$  is the composition of  $\alpha_C$  with the connecting homomorphism  $H^1(\mathcal{N}_{C/S}) \rightarrow H^2(\mathcal{O}_S)$  induced from the exact sequence  $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{N}_{C/S} \rightarrow 0$ , cf. [5, Thm. 1] and [21, Sect. 4] for some details and compare with [15, Ex. 10.6].

We will now prove a theorem which via Remark 2.3 implies Proposition 1.2 by letting  $C = E$ . Note that Theorem 2.2 immediately gives us a formula for  $h^1(\mathcal{N}_C)$  because  $h^1(\mathcal{N}_C) = \dim W - 4d$ .

**Theorem 2.2.** *Let  $W'$  be an irreducible component of  $D(d, g; s)_{sc}$  and let  $W := \text{pr}_1(W') \subseteq H(d, g)_{sc}$  (i.e. if  $d > s^2$ , this means that  $W$  is an  $s$ -maximal family of  $H(d, g)_{sc}$ , in which case  $u := h^0(\mathcal{I}_{C/S}(s)) + h^0(\mathcal{I}_{C/S}(s-4))$  below vanishes). Let  $(C, S)$  be a member of  $W'$  such that  $S$  is smooth of degree  $s$  and*

$$H^1(\mathcal{I}_C(s)) = H^1(\mathcal{I}_C(s-4)) = 0 .$$

*Let  $E$  be a curve on  $S$ ,  $H$  a hyperplane section and suppose  $C \equiv eE + fH$  for some  $e, f \in \mathbb{Z}$ . If  $E$  is ACM, or more generally if  $E$  is unobstructed and satisfies  $H^1(\mathcal{I}_E(s)) = H^1(\mathcal{I}_E(s-4)) = 0$ , then  $W$  is a generically smooth irreducible component of  $H(d, g)_{sc}$  (indeed  $C$  is unobstructed) of dimension*

$$(4-s)d + g + \binom{s+3}{3} - 2 - u + h^0(\mathcal{I}_{E/S}(s-4)) + t \quad \text{where } t := h^1(\mathcal{N}_E) - h^1(\mathcal{O}_E(s)) \geq 0 \quad (11)$$

*if  $e \neq 0$ ; if  $e = 0$  then replace  $h^0(\mathcal{I}_{E/S}(s-4)) + t$  by  $\binom{s-1}{3}$  in (11).*

**Remark 2.3.** *If  $E$  is any curve on a smooth surface  $S$  of degree  $s$  satisfying  $h^1(\mathcal{N}_E) = h^1(\mathcal{O}_E(s))$  and  $H^1(\mathcal{I}_E(s)) = 0$ , then the obstruction group  $\text{coker } \alpha_E = 0$  by (4), whence  $E$  is unobstructed by (6). The unobstructedness assumption on  $E$  in Theorem 2.2 is therefore fulfilled in this case.*

*Proof.* If  $E$  is ACM then  $E$  is unobstructed by [4]. Now since  $H^1(\mathcal{I}_E(s)) = 0$  it follows from the text accompanying (6) that  $D(d(E), g(E); s)$  is smooth at  $(E, S)$ , whence  $\text{Pic}$  is smooth at  $(\mathcal{O}_S(E), S)$  by [8, Rem. 4.5] and  $H^1(\mathcal{I}_E(s-4)) = 0$ . Then  $\text{Pic}$  is smooth at  $(\mathcal{O}_S(C), S)$  because the local rings  $\mathcal{O}_{\text{Pic}, (\mathcal{O}_S(C), S)}$  and  $\mathcal{O}_{\text{Pic}, (\mathcal{O}_S(E), S)}$  are isomorphic, at least up to completion. Indeed by [5, Prop. 2 and Constr. 2],  $\alpha_{\mathcal{O}_S(C)} = e \cdot \alpha_{\mathcal{O}_S(E)}$  and then (10) shows that the morphism between the local deformation functors of  $\text{Pic}$  at  $(\mathcal{O}_S(E), S)$  and  $\text{Pic}$  at  $(\mathcal{O}_S(C), S)$  (induced by  $\mathcal{O}_S(E) \mapsto \mathcal{O}_S(E)^{\otimes a}(f)$ ) is an isomorphism. Hence  $D(d, g; s)$  is smooth at  $(C, S)$  by  $H^1(\mathcal{I}_C(s-4)) = 0$ . Then the smoothness of  $\text{pr}_1$ , due to  $H^1(\mathcal{I}_C(s)) = 0$ , shows that  $W$  is a generically smooth irreducible component of  $H(d, g)_{sc}$ .

To find  $\dim W' = \dim A^1$  and hence  $\dim W = \dim W' - h^0(\mathcal{I}_{C/S}(s))$ , we need to determine  $\dim \text{coker } \alpha_C$ . Using  $H^1(\mathcal{I}_C(s-4)) = 0$  we get that the map  $H^1(\mathcal{N}_{C/S}) \simeq H^0(\mathcal{O}_C(s-4))^\vee \rightarrow H^2(\mathcal{O}_S) \simeq H^0(\mathcal{O}_S(s-4))^\vee$  is injective with cokernel  $H^0(\mathcal{I}_{C/S}(s-4))^\vee$ . It follows that there is an exact sequence

$$0 \longrightarrow \text{coker } \alpha_C \longrightarrow \text{coker } \alpha_{\mathcal{O}_S(C)} \longrightarrow H^0(\mathcal{I}_{C/S}(s-4))^\vee \longrightarrow 0 .$$

Since  $H^1(\mathcal{I}_E(s-4)) = 0$  there is a corresponding exact sequence replacing  $C$  by  $E$ , and the middle term in these sequences are isomorphic because  $\alpha_{\mathcal{O}_S(C)} = e \cdot \alpha_{\mathcal{O}_S(E)}$ . Thanks to (4), we have  $\text{coker } \alpha_E = t \geq 0$  and we obtain the dimension of  $W'$  from the dimension formula accompanying (4). If  $e = 0$ , then  $C$  is a c.i. and we get  $H^1(\mathcal{N}_C) \simeq H^1(\mathcal{O}_C(s)) \oplus H^1(\mathcal{O}_C(f))$ . Using (4), we see that  $\text{coker } \alpha_C \simeq H^1(\mathcal{O}_C(f))$  and we conclude by  $H^1(\mathcal{O}_C(f)) \simeq H^0(\mathcal{O}_C(s-4))^\vee$ .  $\square$

**Remark 2.4.** If  $H^1(\mathcal{N}_E) = 0$ , e.g.  $H^1(\mathcal{O}_E(1)) = 0$  and  $E$  reduced, then  $E$  is unobstructed and  $t = 0$  in Theorem 2.2. Observe, however, that in many non-trivial cases where the unobstructedness of  $E$  is known, there is also a dimension formula of  $h^1(\mathcal{N}_E)$ , making the number  $t$  of Theorem 2.2 explicit, see e.g. [22, Thm. 1.1] for a formula covering both the ACM and Buchsbaum diameter-1 case.

**Remark 2.5.** Using [33, Cor. 4, p. 222], or [2, Prop. 3.4], and, say, the smoothness of  $\pi$  restricted to  $U \cap D(d, g; 4)_{sc}$ , we get that a closed irreducible subset  $W$  of  $H(d, g)_{sc}$ ,  $d > 16$ , whose very general member  $C$  sits on a smooth quartic surface  $S$  with Picard number  $\rho$ , will satisfy  $\dim W \leq g + 35 - \rho$ . Hence if  $W \subset H(d, g)_{sc}$  is 4-maximal (e.g. a component), then  $\rho = 2$  (or  $\rho = 1$  in the c.i. case).

One should compare Remark 2.5 with the following result which is a special case [24, Cor. II 3.8] of a theorem of A. Lopez, see [24, Thm. II 3.1] for a proof.

**Lemma 2.6.** Let  $E \subset \mathbb{P}^3$  be a smooth irreducible curve, let  $n \geq 4$  be an integer and suppose the degree of every minimal generator of the homogeneous ideal of  $E$  is at most  $n - 1$ . Let  $S$  be a very general smooth surface of degree  $n$  containing  $E$  and let  $H$  be a hyperplane section. Then  $\text{Pic}(S) \simeq \mathbb{Z}^{\oplus 2}$  and we may take  $\{\mathcal{O}_S(H), \mathcal{O}_S(E)\}$  as a  $\mathbb{Z}$ -basis for  $\text{Pic}(S)$ .

Finally we recall the definition of  $G(d, s)$ ; the maximum genus of smooth connected curves of degree  $d$  not contained in a surface of degree  $s - 1$ , cf. [12]. By definition,

$$G(d, s) = \max\{g = g(C) \mid (C) \in H(d, g)_{sc} \text{ and } H^0(\mathcal{I}_C(s - 1)) = 0\} \quad (12)$$

In the case where  $d > s(s - 1)$ , Gruson and Peskine showed in [12] that

$$G(d, s) = 1 + \frac{d}{2} \left( \frac{d}{s} + s - 4 \right) - \frac{r(s - r)(s - 1)}{2s} \quad \text{where } d + r \equiv 0 \pmod{s} \text{ for } 0 \leq r < s, \quad (13)$$

and that  $g(C) = G(d, s)$  if and only if  $C$  is linked to a plane curve of degree  $r$  by a c.i. of type  $(s, f)$ ,  $f := (d + r)/s$ . Note that this description of a curve  $C$  of  $H(d, G(d, s))_{sc}$  makes it possible to use Theorem 2.2 to find  $\dim V$  where  $V \subset H(d, G(d, s))_{sc}$  is the irreducible component containing  $(C)$ . Indeed, we may assume a general enough member  $C'$  of  $V$  is contained in a smooth surface of degree  $s$  because the inequality  $r < s$  allows us to start with a smooth plane curve  $E$  of degree  $r$  contained in a smooth surface of degree  $s$  and then make a linkage via a c.i. of type  $(s, f)$  to get  $C'$ . Since  $C' \equiv fH - E$ ,  $H$  a hyperplane section, Theorem 2.2 applies and we get  $\dim V$  from (11).

### 3 Irreducible components of $H(d, g)_{sc}$

In the background section we noticed that the assumption  $H^1(\mathcal{I}_C(s)) = 0$  for  $s = 4$  implies that 4-maximal subsets form generically smooth irreducible components of  $H(d, g)_{sc}$ . We are now looking for a converse, i.e. that  $H^1(\mathcal{I}_C(s)) \neq 0$  implies that maximal subsets form non-reduced components of  $H(d, g)_{sc}$ . If  $s = 3$  this is essentially a conjecture stated in the Introduction. In this section we will see that some ideas of [17] generalize to cover the case  $s > 3$  as well. Indeed, we will show the following result which, together with (13), will be used for proving the results of this paper.

**Theorem 3.1.** Let  $W \subset H(d, g)_{sc}$  be a 4-maximal family whose general member  $C$  is contained in a smooth surface  $S$  of degree 4, and suppose that  $C$  is not a complete intersection of  $S$  and some other surface. If  $h^1(\mathcal{I}_C(1)) \leq d - 25$  and

$$d \geq 31 \quad \text{and} \quad g > 21 + \frac{d^2}{10},$$

then  $W$  is an irreducible component of  $H(d, g)_{sc}$ . Moreover  $W$  is non-reduced if and only if

$$H^1(\mathcal{I}_C(4)) \neq 0.$$

**Remark 3.2.** Let  $C$  be a curve contained in a smooth quartic surface  $S$ . Then it is easy to see that

$$H^1(\mathcal{I}_C(4)) \simeq H^1(\mathcal{I}_{C/S}(4)) , \quad H^1(\mathcal{O}_C(4)) \simeq H^2(\mathcal{I}_C(4)) \simeq H^2(\mathcal{I}_{C/S}(4))$$

and  $H^i(\mathcal{I}_{C/S}(4))^\vee \simeq H^{2-i}(\mathcal{O}_S(C - 4H))$  where  $H$  is a hyperplane section. So to explicitly find non-reduced components given by Theorem 3.1, one should look for curves  $C$  on  $S$  such that the linear system  $|C - 4H|$  is non-empty and contains fixed components.

To prove the theorem we will need

**Proposition 3.3.** Let  $F$  be an integral surface in  $\mathbb{P}^3$  of degree  $s \geq 4$ , let  $S \rightarrow F$  be a desingularization and let  $C$  be a smooth connected curve of degree  $d$  and genus  $g$  such that  $\text{Sing}(F) \cap C$  is a finite set. If  $\mathcal{N}_{C/S}$  is the normal sheaf of  $C \hookrightarrow S$  (i.e. of the proper transform of  $C \hookrightarrow F$ ) and  $\text{Hilb}(F)$  is the Hilbert scheme of curves on  $F$ , then

$$\dim_{(C)} \text{Hilb}(F) \leq \dim H^0(\mathcal{N}_{C/S}) \leq \max \left\{ \frac{d^2}{s} - g + 1, \frac{d^2}{2s} + 1 \right\}.$$

*Proof.* Indeed the proof is as in [17], Lemmas 22 and 23 where we considered the two following cases; (1)  $H^1(\mathcal{N}_{C/S}) = 0$  in which we used Riemann-Roch, and (2)  $H^0(\mathcal{N}_{C/S}) \neq 0$ ,  $H^1(\mathcal{N}_{C/S}) \neq 0$  in which we used Clifford's theorem. We also needed Hodge's index theorem to get  $\deg \mathcal{N}_{C/S} = C^2 \leq d^2/s$ . Combining, we easily get the proposition.  $\square$

The following proposition was mainly proved in [17, Prop. 20] in the case  $s(C) = 4$  (where the weak assumption, " $\text{Sing}(F) \cap C$  is finite" or  $d > (s(C) - 1)^2$ , seems missing). To prove Theorem 3.1 we need, however, the result for  $s(C) = 5$ .

**Proposition 3.4.** Let  $V$  be an irreducible component of  $\text{H}(d, g)_{sc}$  whose general curve  $C$  sits on some integral surface  $F$  of degree  $s \geq 4$ . If  $d > s^2$ , then

$$\dim V \leq \binom{s+3}{3} - 1 + \max \left\{ \frac{d^2}{s} - g, \frac{d^2}{2s}, (4-s)d + g - 1 + h^0(\mathcal{O}_C(s-4)) \right\}.$$

*Proof.* Let  $W$  be any irreducible component of  $\text{D}(d, g; s)_{sc}$  containing  $(C, F)$ . Since the  $2^{\text{nd}}$  projection,  $pr_2 : \text{D}(d, g; s)_{sc} \rightarrow \text{H}(s)$  has the Hilbert scheme  $\text{Hilb}(F)$  as its fiber over  $(F)$ , it follows that

$$\dim W \leq \dim pr_2(W) + \dim_{(C)} \text{Hilb}(F). \quad (14)$$

Suppose  $F$  is smooth. Then  $\dim pr_2(W) \leq \dim_{(F)} \text{H}(s) = \binom{s+3}{3} - 1$ . Moreover,  $\mathcal{N}_{C/F} \simeq \omega_C \otimes \omega_F^{-1}$  leads to  $\chi(\mathcal{N}_{C/F}) = \chi(\omega_C(4-s)) = (4-s)d + g - 1$  and

$$\dim_{(C)} \text{Hilb}(F) \leq h^0(\mathcal{N}_{C/F}) = (4-s)d + g - 1 + h^0(\mathcal{O}_C(s-4)).$$

Suppose  $F$  is not smooth, but integral, then  $pr_2$  is at least non-dominating, whence  $\dim pr_2(W) \leq \binom{s+3}{3} - 2$ . To use Proposition 3.3 to bound  $\dim_{(C)} \text{Hilb}(F)$ , we must show that  $\text{Sing}(F) \cap C$  is a finite set. Indeed if this set is not finite, then the smooth connected curve  $C$  is contained in  $\text{Sing}(F)$  which implies  $d \leq (s-1)^2$  because there is a c.i. of type  $(s-1, s-1)$  containing  $C$  (chosen among the partial derivatives of the form defining  $F$ ). This contradicts an assumption of Proposition 3.4 while the other assumptions imply the existence of an irreducible component  $W \ni (C, F)$  of  $\text{D}(d, g; s)_{sc}$  which dominates  $V$  under the first projection  $pr_1$  given in (5). Since  $d > s^2$  then  $\dim V = \dim W$  and we can use (14) and the upper bounds of  $\dim_{(C)} \text{Hilb}(F)$  to get Proposition 3.4.  $\square$

*Proof (of Theorem 3.1).* To see that  $W$  is an irreducible component, we suppose there exists a component  $V$  of  $H(d, g)_{sc}$  satisfying  $W \subset V$  and  $\dim W < \dim V$ . Then  $s := s(V) \geq 5$  by the definition of a 4-maximal family. Moreover  $s = 5$  since the case  $s \geq 6$  can be excluded because the genus turns out to satisfy  $g > G(d, 6)$  by the assumptions of the theorem and by using (12) and (13). To get a contradiction we will use Proposition 3.4 for  $s = 5$  and that  $\dim W = g + 33$ , cf. the background section. Let  $C'$  be the general curve of  $V$ . Then  $s(C') = 5$  and  $C'$  is a smooth connected curve. It follows that a surface  $F'$  containing  $C'$  of the least possible degree, namely 5, is integral. We get

$$g + 33 < 55 + \max \left\{ \left\lfloor \frac{d^2}{5} \right\rfloor - g, \left\lfloor \frac{d^2}{10} \right\rfloor, -d + g - 1 + 4 + h^1(\mathcal{I}_{C'}(1)) \right\}.$$

Suppose the maximum to the right is obtained by  $\lfloor d^2/5 \rfloor - g$ . Then since  $g + 33 < 55 + d^2/5 - g$  is equivalent to  $g < 11 + d^2/10$ , we get a contradiction to the displayed assumption of the theorem. Similarly,  $g + 33 < 55 + \lfloor d^2/10 \rfloor$  will lead to a contradiction. Finally if we suppose

$$g + 33 < 55 - d + g - 1 + 4 + h^1(\mathcal{I}_{C'}(1)),$$

i.e.  $h^1(\mathcal{I}_{C'}(1)) > d - 25$  and we use that  $h^1(\mathcal{I}_{C'}(1)) \leq h^1(\mathcal{I}_C(1))$  by semi-continuity, we get  $h^1(\mathcal{I}_C(1)) > d - 25$  which again is a contradiction to the assumptions. Thus we have proved that  $W$  is an irreducible component of  $H(d, g)_{sc}$ .

Now using (7), i.e.  $\dim W + h^1(\mathcal{I}_C(4)) = h^0(\mathcal{N}_C)$ , then it is straightforward to get the final statement of the theorem, and we are done.  $\square$

**Corollary 3.5.** *With notations and assumptions as in the first sentence in Theorem 3.1, suppose in addition*

$$g > \min \left\{ G(d, 5) - 1, \frac{d^2}{10} + 21 \right\} \quad \text{and} \quad d \geq 21.$$

*Then  $W$  is an irreducible component of  $H(d, g)_{sc}$ , and  $W$  is non-reduced if and only if  $H^1(\mathcal{I}_C(4)) \neq 0$ .*

*Proof.* One checks that the minimum value in the corollary is equal to  $G(d, 5) - 1$  (resp.  $21 + d^2/10$ ) for  $21 \leq d \leq 44$  (resp.  $d \geq 45$ ). Moreover,  $h^1(\mathcal{I}_C(1)) = h^0(\mathcal{O}_C(1)) - 4 \leq \max\{d - g, \frac{d}{2}\} - 3$  for a non-plane curve by Clifford's theorem and Riemann-Roch. Hence if  $d \geq 45$ , we get  $h^1(\mathcal{I}_C(1)) \leq d - 25$  and we conclude by Theorem 3.1.

If  $21 \leq d \leq 44$  we suppose there is an irreducible component  $V$  of  $H(d, g)_{sc}$  satisfying  $W \subset V$  and  $\dim W < \dim V$ . We may suppose either  $g > G(d, 5)$  or  $g = G(d, 5)$ . In the first case we get  $s(V) = 4$  which contradicts the 4-maximality of  $W$ . In the remaining case we have  $\dim V \leq 58 - d + g$  for  $d \geq 25$  by (11) which contradicts  $g + 33 = \dim W < \dim V$ . For  $21 \leq d \leq 24$  we get  $\dim V \leq 56 - d + g - h^0(\mathcal{I}_{C/S}(s))$  by (11), whence  $\dim V \leq g + 33$ , implying a contradiction. Finally using (7) we easily get the statement on non-reducedness of the corollary, and we are done.  $\square$

**Remark 3.6.** *Let  $C$  be a general curve of a maximal family  $W$ . The analogue of Theorem 3.1 for  $s(C) = 3$  states that  $W$  is an irreducible (resp. and non-reduced) component of  $H(d, g)_{sc}$  provided*

$$g > 7 + (d - 2)^2/8 \quad \text{and} \quad d \geq 27 \quad (\text{resp. } g > 7 + (d - 2)^2/8, \quad d \geq 18 \quad \text{and} \quad H^1(\mathcal{I}_C(3)) \neq 0),$$

*as one may deduce from [17]. To show that  $W$  is a non-reduced irreducible component, the above result turned out to be quite useful in [17]. This result, together with Theorem 3.1 for  $s(C) = 4$ , improve upon what we may show by only using (13) by  $k + d/2$ ,  $k$  a constant, cf. Corollary 3.5. This improvement is not necessary for Theorem 1.1 of this paper because the curves in II) satisfy  $g > G(d, 5) - 1$ . We need, however, Theorem 3.1 in Section 5, and we hope it applies to other classes of components where  $s(C) = 4$ , as it did for  $s(C) = 3$ . If we try to generalize Theorem 3.1 to  $s(C) \geq 5$ , we unfortunately get nothing more than what (13) implies.*



## 4 Components of $H(d, g)_{sc}$ for $s = 4$

In this section we prove the theorem on quartic surfaces stated in the Introduction. Note that the existence of curves and surfaces as described in Theorem 1.1 follows from the results of J.C. Ottem of the appendix, see also [10] for a study of curves on smooth quartic surfaces.

*Proof of Theorem 1.1.* We get  $d = a + 3b$ ,  $g = 3ab - a^2 + 1$  from  $d = C \cdot H$ ,  $g = 1 + C^2/2$  and since  $C \notin |nH|$  for every  $n \in \mathbb{Z}$  by assumption, it follows that  $C$  is not a c.i. in  $S$ . By [17, Thm. 10 and Lem. 13], see Section 2 of this paper, the Hilbert-flag scheme  $D(d, g; s)$  for  $s = 4$  is smooth at  $(C, S)$  of dimension  $\dim A^1 = g + 33$ . Hence  $(C, S)$  belongs to a unique irreducible component of  $D(d, g; 4)_{sc}$  whose image under the  $1^{st}$  projection,  $pr_1 : D(d, g; 4)_{sc} \rightarrow H(d, g)_{sc}$ , is the 4-maximal subset  $W$  of the theorem because the assumption  $d > 16$  implies  $H^0(\mathcal{I}_C(3)) \simeq H^0(\mathcal{O}_S(3H - C)) = 0$ , i.e.  $s(W) = 4$ . To show all the properties of  $W$  appearing in Theorem 1.1, we consider the rational map

$$\pi : D(d, g; 4)_{sc} \dashrightarrow \text{Pic}$$

onto the relative Picard scheme induced by  $\pi((C_1, S_1)) = (\mathcal{O}_{S_1}(C_1), S_1)$  as in (9). Thanks to [8, Rem. 4.5], one knows that  $\pi$  at  $(C, S)$ , hence  $\text{Pic}$  at  $(\mathcal{O}_S(C), S)$ , is smooth because  $H^1(S, \mathcal{O}_S(C)) \simeq H^1(\mathbb{P}^3, \mathcal{I}_C)^\vee = 0$ . Since the fiber  $\pi^{-1}((\mathcal{O}_S(C), S))$  is  $g$ -dimensional, the dimension of the image of  $\pi$  in some neighborhood of  $(\mathcal{O}_S(C), S)$  in  $\text{Pic}$  is 33. By the same argument the rational map  $\pi' : D(1, 0; 4)_{sc} \dashrightarrow \text{Pic}$  induced by  $\pi'((E_1, S_1)) = (\mathcal{O}_{S_1}(E_1), S_1)$ ,  $E_1$  a line on a smooth quartic  $S_1$ , is an isomorphism in some smooth neighborhood  $U'$  of  $(E, S)$ ,  $E := \Gamma_1$ , of dimension 33 because  $\pi'$  is injective in  $U'$ ,  $h^0(S, \mathcal{O}_S(E)) = 1$  and  $H^1(\mathbb{P}^3, \mathcal{I}_E) = 0$ , cf. [15, Ex. 6.10]. Note that  $a\Gamma_1 + b\Gamma_2 = (a - b)E + bH$  and that there is a morphism

$$\eta : U \rightarrow \text{Pic} \quad \text{induced by} \quad (\mathcal{O}_{S_1}(E_1), S_1) \mapsto (\mathcal{O}_{S_1}(E_1)^{\otimes(a-b)}(b), S_1)$$

in some neighborhood  $U \subset \pi'(U')$  of  $(\mathcal{O}_S(E), S)$  in  $\text{Pic}$ . Using the arguments in the first paragraph of the proof of Theorem 2.2, we may suppose that  $\eta$  is an isomorphism onto its image. Now since  $d > 16$  there is only one quartic surface  $\tilde{S}$  containing a very general member of  $W$ . Recalling that the smooth morphism  $\pi$  (i.e. smooth in some neighborhood of  $(C, S)$ ) maps generic points onto generic points and that  $D(1, 0; 4)_{sc}$  is irreducible ( $pr_1 : D(1, 0; 4)_{sc} \rightarrow H(1, 0)_{sc}$  is irreducible by [19, Thm. 1.16]), it follows from Lemma 2.6 and the isomorphism  $\eta$  that  $\tilde{S}$  allows a  $\mathbb{Z}$ -basis of  $\text{Pic}(\tilde{S})$  consisting of the classes a line  $\tilde{E}$  and a hyperplane section. We also get that  $W$  is unique in the sense that every  $C \subset S$  as in the theorem belongs to the same  $W$  because the fiber  $\pi^{-1}((\mathcal{O}_{S_1}(C_1), S_1))$  is given by the linear system  $|C_1| \cap D(d, g; 4)_{sc}$  which is irreducible, whence  $\pi^{-1}(\eta(U))$  is irreducible (cf. [16, Prop. 1.8]). Hence we have all the stated properties of  $W$  provided we can take a  $\mathbb{Z}$ -basis of  $\text{Pic}(\tilde{S})$  as described in the theorem. This is straightforward. Indeed there is a hyperplane section  $\tilde{H}$  of  $\tilde{S}$  containing  $\tilde{E}$  such that  $\Gamma := \tilde{H} - \tilde{E}$  is a smooth curve of degree 3 ([32]) and instead of the basis  $\{\mathcal{O}_S(\tilde{E}), \mathcal{O}_S(\tilde{H})\}$  given by Lemma 2.6, we may take the classes of  $\{\tilde{E}, \Gamma\}$  as a  $\mathbb{Z}$ -basis of  $\text{Pic}(\tilde{S})$ .

For the rest of the proof we use (19) of the appendix, (1) as in Corollary 3.5, Remark 3.2 to see  $h^1(I_C(4)) = h^1(\mathcal{O}_S(C - 4H))$  and Proposition 1.2.

I) It suffices to show that  $H^1(S, \mathcal{O}_S(C - 4H)) = 0$  by Proposition 1.2. But this is immediate from (19) since the inequality  $4 < a < \frac{3b}{2} - 1$  implies  $a - 4 > 0$  and  $2(a - 4) \leq 3(b - 4) + 1$ .

II) By Corollary 3.5 it suffices to show that  $H^1(S, \mathcal{O}_S(C - 4H)) \neq 0$  for the classes in (2) of Theorem 1.1. Also this is immediate by (19), since the inequalities in (2) and  $d > 16$  imply  $2(a - 4) > 3(b - 4) + 1$  and  $a - 4 > 0$ . The lattice points in this region satisfying (1) are then found by inspection.  $\square$

**Remark 4.1.** Note that if  $D := C - 4H = a\Gamma_1 + b\Gamma_2$  is effective and  $h^1(S, \mathcal{O}_S(D)) > 0$ , then either  $\Gamma_1$  is a fixed component of  $|D|$  or  $D$  is composed with a pencil, in which case  $C = 4\Gamma_1 + r\Gamma_2$  for  $r \geq 5$ . In the latter case, it is easy to verify that  $C$  does not satisfy the constraints (1), hence does not lead to non-reduced components by the theory we have so far.

**Remark 4.2.** For the non-reduced components in Theorem 1.1 the dimension of the tangent space of  $H(d, g)_{sc}$  at a general  $(C)$  of  $W$  is equal to  $g + 33 + h^1(I_C(4))$  by (7). We claim that  $h^1(I_C(4)) = 1$  (resp.  $h^1(I_C(4)) = 2$ ,  $h^1(I_C(4)) = 4$ ) for the family  $a$  (resp.  $b$ ),  $c$ ). To see it we use the short exact sequence in the proof of Lemma 6.1 in the appendix for  $\Gamma := \Gamma_1$  and  $D := C - 4H$  (and then to  $D := C - 4H - \Gamma_1$  for the class  $b$ ). Taking cohomology and counting dimensions, we get the claim.

#### 4.1 Other quartic surfaces

As in Theorem 1.1 we expect that there are many other smooth quartic surfaces, sufficiently general in the Noether-Lefschetz locus, where one may explicitly describe classes of non-reduced, and generically smooth, components of  $H(d, g)_{sc}$ . Our method so far uses results from the background section and Corollary 3.5 where the latter puts severe restrictions on the genus. To illustrate this we consider the following smooth quartic surface where the classes of two smooth conics  $\{\Gamma_1, \Gamma_2\}$  is a basis of  $\text{Pic}(S)$ ,  $\begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$  is the intersection matrix and  $H = \Gamma_1 + \Gamma_2$  a hyperplane section, cf. the appendix for existence. Using Proposition 6.4 one may see precisely when  $H^1(\mathcal{I}_C(4)) = 0$ , and we get at least:

**Proposition 4.3.** Let  $S \subset \mathbb{P}^3$  be a smooth quartic surface with  $\Gamma_1, \Gamma_2$  as above, let  $C \equiv (a, b)$  be a smooth connected curve and suppose  $a \neq b$  and  $d > 16$ . Then  $C$  belongs to a unique 4-maximal family  $W \subseteq H(d, g)_{sc}$ . Moreover if  $\tilde{S}$  is a quartic surface containing a very general member of  $W$ , then  $\text{Pic}(\tilde{S})$  is freely generated by the classes of two rational conics, and every  $C \equiv (a, b)$  contained in some surface  $S$  as above belongs to  $W$ . Furthermore  $\dim W = g + 33$ ,

$$d = 2a + 2b, \quad g = 4ab - a^2 - b^2 + 1 \quad \text{and}$$

$W$  is a generically smooth, irreducible component of  $H(d, g)_{sc}$  provided  $\frac{b}{2} + 2 \leq a \leq 2b - 4$ .

*Proof.* The proof of the properties of the maximal family  $W$  follows as in the first part of Theorem 1.1. The remaining part follows from Proposition 1.2 and Proposition 6.4 since  $H^1(S, \mathcal{O}_S(C - 4H)) = 0$  if and only if  $C - 4H$  is nef if and only if  $\frac{b+4}{2} \leq a \leq 2(b-2)$ .  $\square$

**Remark 4.4.** We may by symmetry restrict the range of Proposition 4.3 to  $a > b$ . Then there are 4 families in the range  $2b - 4 < a \leq 2b$  which satisfy  $H^1(\mathcal{I}_C(4)) \neq 0$ . They are of the form  $(5 + 2k, 4 + k)$   $(6 + 2k, 4 + k)$   $(7 + 2k, 4 + k)$   $(8 + 2k, 4 + k)$ ,  $k \geq 1$ . Unfortunately, (1) does not hold for any of these classes, so we can not conclude that they correspond to non-reduced components by the results we have so far. We expect, however, that they are non-reduced components.

### 5 Non-reduced components of $H(d, g)_{sc}$ for $s = 3$

In this section we look at progresses to the conjecture below. Note that a maximal family  $W$  is closed and irreducible by our definition, and that  $\dim W = d + g + 18$  always holds provided  $d > 9$ .

**Conjecture 5.1.** Let  $W$  be a maximal family of smooth connected, linearly normal space curves of degree  $d$  and genus  $g$ , whose general member  $C$  is contained in a smooth cubic surface. Then  $W$  is a non-reduced irreducible component of  $H(d, g)_{sc}$  if and only if

$$d \geq 14, \quad 3d - 18 \leq g \leq (d^2 - 4)/8 \quad \text{and} \quad H^1(\mathcal{I}_C(3)) \neq 0.$$

This conjecture, originating in [18], is here presented by modifications proposed by Ellia [3] (see also [1] by Dolcetti, Pareshi), because they found counterexamples which heavily depended on the fact the general curves were *not* linearly normal (i.e. the curves satisfied  $H^1(\mathcal{I}_C(1)) \neq 0$ ).

The conjecture is known to be true in many cases. Indeed Mumford's well known example ([29]) of a non-reduced component is in the range of Conjecture 5.1 ("minimal with respect to both degree and genus"). Also the main result by the author in [17] shows that the conjecture holds provided  $g > 7 + (d - 2)^2/8$ ,  $d \geq 18$ , and Ellia makes further progresses in [3] which we comment on later. Recently Nasu proves (and reproves) a part of the conjecture by showing that the cup-product

$$H^0(\mathcal{N}_C) \times H^0(\mathcal{N}_C) \rightarrow H^1(\mathcal{N}_C)$$

is nonzero if  $h^1(\mathcal{I}_C(3)) = 1$  ([30]). In this section we will see that the methods of [17] and a nice result of Ellia in [3] imply that we can extend the range where Conjecture 5.1 holds a lot.

Now recall that a smooth cubic surface  $S$  is obtained by blowing up  $\mathbb{P}^2$  in six general points (see [14] and [13]). Taking the linear equivalence classes of the inverse image of a line in  $\mathbb{P}^2$  and  $-E_i$  (minus the exceptional divisors),  $i = 1, \dots, 6$ , as a basis for  $\text{Pic}(S)$ , we can associate a curve  $C$  on  $S$  and its corresponding invertible sheaf  $\mathcal{O}_S(C)$  with a 7-tuple of integers  $(\delta, m_1, \dots, m_6)$  satisfying

$$\delta \geq m_1 \geq \dots \geq m_6 \quad \text{and} \quad \delta \geq m_1 + m_2 + m_3. \quad (15)$$

The degree and the (arithmetic) genus of the curve are given by

$$d = 3\delta - \sum_{i=1}^6 m_i, \quad g = \binom{\delta-1}{2} - \sum_{i=1}^6 \binom{m_i}{2}.$$

In terms of a 7-tuple  $(\delta, m_1, \dots, m_6)$  satisfying (15) one may use Kodaira vanishing theorem and a further analysis (see [17, Lem. 16 and Cor. 17]) to verify the following facts for a curve  $C$ ;

(A) If  $m_6 \geq 3$  and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 9, \lambda + 3, 3, \dots, 3)$  for any  $\lambda \geq 2$ , then  $H^1(\mathcal{I}_C(3)) = 0$ . In particular if a curve on a smooth cubic satisfies  $g > (d^2 - 4)/8$ , then

$$H^1(\mathcal{I}_C(3)) = 0.$$

(B) If  $m_6 \geq 1$  and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 3, \lambda + 1, 1, \dots, 1)$  for any  $\lambda \geq 2$ , then  $H^1(\mathcal{I}_C(1)) = 0$ . Moreover, in the range  $d \geq 14$  and  $g \geq 3d - 18$ , we have

$$H^1(\mathcal{I}_C(3)) \neq 0 \text{ and } H^1(\mathcal{I}_C(1)) = 0 \quad \text{if and only if} \quad 1 \leq m_6 \leq 2.$$

**Remark 5.2.** *i) The explicit size of the interval where  $H_*^1(\mathcal{I}_C) := \bigoplus_v H^1(\mathcal{I}_C(v))$  is non-vanishing (and a proof of it) was originally found by Peskine and Gruson (see [18, Prop. 3.1.3]).*

*ii) The case  $m_6 = 0$  is treated by Dolcetti and Pareshi in [1]. In this case they found a range in the  $(d, g)$ -plane where the maximal subsets  $W$  were contained in a non-reduced component of dimension  $> d + g + 18$ , see also [3, Rem. VI.6].*

Using (A) and the fact that  $H^1(\mathcal{I}_C(3)) = 0$  implies unobstructedness and  $\dim W = d + g + 18$  (for  $d > 9$ ), one may easily see that the conditions of Conjecture 5.1 are necessary for  $W$  to be a non-reduced component. The conjecture therefore really deals with the converse, and we may suppose  $m_6 = 1$  or  $2$  by (B). For both values the main theorem of this section shows that the conjecture is true under weak assumptions, thus generalizing the main results of [3] and [17] to:

**Theorem 5.3.** *Let  $W$  be a 3-maximal family of smooth connected space curves, whose general member sits on a smooth cubic surface  $S$  and corresponds to the 7-tuple  $(\delta, m_1, \dots, m_6)$ ,  $\delta \geq m_1 \geq \dots \geq m_6$  and  $\delta \geq m_1 + m_2 + m_3$ , of  $\text{Pic}(S)$ . Then*

*i)  $W$  is a generically smooth, irreducible component of  $H(d, g)_{sc}$  provided*

$$m_6 \geq 3 \text{ and } (\delta, m_1, \dots, m_6) \neq (\lambda + 9, \lambda + 3, 3, \dots, 3) \text{ for any } \lambda \geq 2,$$

*ii)  $W$  is a non-reduced irreducible component of  $H(d, g)_{sc}$  provided;*

*a)  $m_6 = 2, m_5 \geq 4, d \geq 21$  and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 12, \lambda + 4, 4, \dots, 4, 2)$  for any  $\lambda \geq 2$ , or*

*b)  $m_6 = 1, m_5 \geq 6, d \geq 35$  and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 18, \lambda + 6, 6, \dots, 6, 1)$  for any  $\lambda \geq 2$ , or*

*c)  $m_6 = 1, m_5 = 5, m_4 \geq 7, d \geq 35$  and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 21, \lambda + 7, 7, \dots, 7, 5, 1)$  for  $\lambda \geq 2$ .*

In the exceptional case  $(\lambda + 9, \lambda + 3, 3, \dots, 3)$  of *i)* we have  $H^1(\mathcal{O}_C(3)) = 0$ ; whence  $W$  is contained a unique generically smooth irreducible component  $V$  of  $H(d, g)_{sc}$  and  $\dim V - \dim W = h^1(\mathcal{I}_C(3))$  (cf. [17, Thm. 1]).

To prove Theorem 5.3, we will need the following two results:

**Proposition 5.4.** *(Ellia) Let  $d$  and  $g$  be integers such that  $d \geq 21$  and  $g \geq 3d - 18$ , let  $W$  be as in Theorem 5.3 and suppose the general curve  $C$  of  $W$  satisfies  $H^1(\mathcal{I}_C(1)) = 0$ . If  $C'$  is a generization of  $C$  in  $H(d, g)_{sc}$  satisfying  $H^0(\mathcal{I}_{C'}(3)) = 0$ , then  $H^0(\mathcal{I}_{C'}(4)) = 0$ .*

*Proof.* See [3, Prop. VI.2]. □

We remark that Ellia uses this key proposition to prove the conjecture provided  $d \geq 21$  and  $g > G(d, 5)$ , cf. (12). His result is in most cases clearly better than the one in [17] which requires  $g > 7 + (d - 2)^2/8$ ,  $d \geq 18$ , because  $G(d, 5) = d^2/10 + d/2 + \epsilon$ ,  $\epsilon$  a correction term. There is, however, quite a lot of cases where Theorem 5.3 imply the conjecture while this result of Ellia does not.

**Lemma 5.5.** *Let  $C$  be a curve sitting on a smooth cubic surface  $S$ , whose corresponding invertible sheaf is given by  $(\delta, m_1, \dots, m_6)$ ,  $\delta \geq m_1 \geq \dots \geq m_6$  and  $\delta \geq m_1 + m_2 + m_3$ . If  $v$  is a non-negative integer such that  $m_3 \geq v$ , and  $(\delta, m_1, \dots, m_6) \neq (\lambda + 3v, \lambda + v, v, v, m_4, m_5, m_6)$  for any  $\lambda \geq 2$ , then*

$$h^0(\mathcal{I}_C(v)) - h^1(\mathcal{I}_C(v)) \geq \binom{v}{3} - \sum_{m_i < v} \binom{v + 1 - m_i}{2}$$

where the sum is taken among those  $i \in \{4, 5, 6\}$  satisfying  $m_i < v$ .

*Proof.* Let  $b_i := \max\{0, m_i - v\}$  and notice that the invertible sheaf  $\mathcal{L}$ , given by  $(\delta - 3v, b_1, \dots, b_6)$ , is generated by global sections because  $b_6 \geq 0$  and  $\delta - 3v \geq b_1 + b_2 + b_3$  (cf. [13, Sect. 2]). Moreover  $(\delta - 3v, b_1, \dots, b_6) \neq (\lambda, \lambda, 0, \dots, 0)$  for  $\lambda \geq 2$  by assumption; whence  $H^0(\mathcal{L})$  contains a smooth connected curve  $\overline{D}$  (take  $\overline{D} = 0$  in the special case  $(\delta - 3v, b_1, \dots, b_6) = (\lambda, \lambda, 0, \dots, 0)$  with  $\lambda = 0$ ).

Let  $n_i := -\min\{0, m_i - v\}$  for  $i \in \{4, 5, 6\}$ , let  $F := \sum n_i E_i$  and observe that  $D := \overline{D} + F$  is an effective divisor (or zero) of the linear system  $|C - vH|$  corresponding to  $(\delta - 3v, m_1 - v, \dots, m_6 - v)$ . By e.g. the algorithm of [9, Rem. 2.7], for finding the Zariski decomposition, it is clear that  $F$  is the fixed component of  $|D|$ . Now, as in Lem. 2.5 and Cor. 2.6 of [30], taking global sections of the sequence  $0 \rightarrow \mathcal{I}_{C/S}(v) \simeq \mathcal{O}_S(-C + vH) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_D \rightarrow 0$ , we get

$$h^1(\mathcal{I}_C(v)) = h^1(\mathcal{I}_{C/S}(v)) = h^0(\mathcal{O}_D) - 1 = h^0(\mathcal{O}_{\overline{D}}) + h^0(\mathcal{O}_F) - 1 = h^0(\mathcal{O}_F)$$

for  $\overline{D} \neq 0$  because  $\overline{D} \cdot F = 0$  ( $h^1(\mathcal{I}_C(v)) = h^0(\mathcal{O}_F) - 1$  for  $\overline{D} = 0$ ). The lines  $E_i$  are skew and we get  $h^0(\mathcal{O}_F) = \sum h^0(\mathcal{O}_{n_i E_i}) = \sum \binom{n_i + 1}{2}$ . Finally  $h^0(\mathcal{I}_C(v)) = h^0(\mathcal{I}_{C/S}(v)) + \binom{v}{3} \geq \binom{v}{3}$  (equality holds, but we don't need it) and we are done. □

*Proof of Theorem 5.3.* i) is a special case of [17, Thm. 1].

ii) By (7) we get

$$\dim W + h^1(\mathcal{I}_C(3)) = h^0(\mathcal{N}_C) . \quad (16)$$

Since  $h^1(\mathcal{I}_C(3)) \neq 0$ , it suffices to prove that  $W$  is an irreducible *component* of  $H(d, g)_{sc}$  because if it is, then  $\dim W < h^0(\mathcal{N}_C)$  implies that the general curve  $C$  of  $W$  is obstructed, i.e.  $W$  is non-reduced.

a) To get a contradiction, suppose  $W$  is *not* a component. Since  $W$  is a maximal family of curves on a cubic surface, there exists a generization  $C'$  of  $C$  satisfying  $h^0(\mathcal{I}_{C'}(3)) = 0$ . By semi-continuity,  $h^1(\mathcal{O}_{C'}(4)) \leq h^1(\mathcal{O}_C(4))$ . Combining with  $\chi(\mathcal{I}_{C'}(4)) = \chi(\mathcal{I}_C(4))$ , it follows that  $h^0(\mathcal{I}_{C'}(4)) - h^1(\mathcal{I}_{C'}(4)) \geq h^0(\mathcal{I}_C(4)) - h^1(\mathcal{I}_C(4))$ . However, by Lemma 5.5, we have  $h^0(\mathcal{I}_C(4)) - h^1(\mathcal{I}_C(4)) \geq 1$ , hence  $h^0(\mathcal{I}_{C'}(4)) \geq 1$ . Since the curve is linearly normal by (B), this inequality contradicts the conclusion of Proposition 5.4.

b) Again it suffices to prove that  $W$  is an irreducible component of  $H(d, g)_{sc}$ . To get a contradiction we suppose there is a generization  $C'$  of  $C$  satisfying  $h^0(\mathcal{I}_{C'}(3)) = 0$ . By semi-continuity of  $h^1(\mathcal{O}_C(v))$  and Lemma 5.5, we get

$$h^0(\mathcal{I}_{C'}(v)) - h^1(\mathcal{I}_{C'}(v)) \geq h^0(\mathcal{I}_C(v)) - h^1(\mathcal{I}_C(v)) \geq \binom{v}{3} - \binom{v}{2} \text{ for } 1 \leq v \leq 6 .$$

Hence  $h^0(\mathcal{I}_{C'}(6)) - h^1(\mathcal{I}_{C'}(6)) \geq 5$ . Since  $s(C') \geq 5$  by Proposition 5.4 and (B),  $C'$  is contained in a c.i. of bidegree (5,6) or (6,6). Hence  $d \leq 36$  and we have a contradiction except when  $d = 35$  or 36. In the case  $d = 36$ ,  $C'$  is a c.i. satisfying  $h^0(\mathcal{I}_{C'}(6)) \geq 5$ , and if  $d = 35$ , we can link  $C'$  to a line  $D$  satisfying  $h^1(\mathcal{O}_D(2)) \neq 0$  (because  $h^0(\mathcal{I}_{C'}(6)) > 2$ ), i.e. we get a contradiction in both cases, and we are done.

c) The proof is similar to b), remarking only that we now have  $h^0(\mathcal{I}_{C'}(6)) \geq 4$  and  $h^0(\mathcal{I}_{C'}(7)) \geq 11$  by Lemma 5.5, i.e.  $C'$  is contained in a c.i. of bidegree (5,7) or (6,6), and since the case where  $C'$  is a c.i. of bidegree (5,7) can not occur (the dimension of an irreducible component of  $H(d, g)_{sc}$  whose general curve is a c.i. of type (5,7) is much smaller than  $d + g + 18$ ) we conclude as in b).  $\square$

**Remark 5.6.** i) *Theorem 5.3 (without c) of ii)) was lectured at a workshop organized by the "Space Curves group" of Europroj, at the Emile Borel Center, Paris in May 1995, and may be known to some experts in the field (cf. [15, p. 95]), but it has not been published. The appendix in the preprint [20] covers the important results of the talk, and much of the material is included here. Note that we in Lemma 19 of [20] should replace equality by inequality, exactly as we now do in the displayed formula of Lemma 5.5 (we see from its proof that equality almost always holds, except when  $\overline{D} = 0$ ). This correction do no harm to the arguments of Theorem 5.3 since it is precisely the inequality we need in its proof. In the proof of Lemma 5.5 we follow closely corresponding results in [30] which is based on making the fixed component of  $|C - vH|$  explicit. Lemma 5.5 for  $v = 4$  imply Lemma 18 of [17].*

ii) *The case a) of Theorem 5.3 ii) is fully generalized in [30]. Indeed Nasu shows that the cup-product (primary obstruction) of the general curve of any maximal family  $W$  satisfying  $m_6 = 2$  and  $m_5 \geq 3$  is non-vanishing. We think his approach may be adequate for proving the whole conjecture.*

Finally using Proposition 3.4 for  $s = 5$  and closely following the proof of Theorem 3.1 (replacing  $\dim W = g + 33$  by  $\dim W = d + g + 18$  in the argument and noticing that a generization  $C'$  satisfies  $s(C') \geq 5$  by Ellia's Proposition 5.4), we immediately get the following result.

**Proposition 5.7.** *Let  $W$  be a 3-maximal family of smooth connected space curves, whose general member is linearly normal and sits on a smooth cubic surface. If*

$$g > \max \left\{ \frac{d^2}{10} - \frac{d}{2} + 18, G(d, 6) \right\}, \quad d \geq 31, \quad (17)$$

then  $W$  is an irreducible component of  $H(d, g)_{sc}$ . Moreover,  $W$  is non-reduced if and only if  $H^1(\mathcal{I}_C(3)) \neq 0$ . In particular Conjecture 5.1 holds in the range (17).

Note that we in (17) have  $G(d, 6) \geq \frac{d^2}{10} - \frac{d}{2} + 18$  if and only if  $d \leq 74$ . We can weaken the assumption  $g > G(d, 6)$  by using Proposition 3.4 also for  $s = 6$  and 7. Indeed for any  $t$  such that  $6 \leq t \leq 8$  we can conclude as in Proposition 5.7 provided  $g > \max \{ \frac{d^2}{10} - \frac{d}{2} + 18, G(d, t) \}$ ,  $d > t(t-1)$ . Since  $G(d, 8) \leq \frac{d^2}{10} - \frac{d}{2} + 18$  for  $d \geq 58$ , we obtain all conclusions of Proposition 5.7 in the range

$$g > \frac{d^2}{10} - \frac{d}{2} + 18, \quad d \geq 58. \quad (18)$$

## 6 Appendix: Quartic surfaces with Picard number two

by John C. Ottem

The purpose of this appendix is to construct the two quartic surfaces needed in J.O. Kleppe's paper and to give the necessary computations needed in the proof of Theorem 1.1 and Proposition 4.3.

A smooth quartic surface  $S \subset \mathbb{P}^3$  is the standard example of a projective K3 surface, i.e., a projective surface with trivial canonical bundle  $K_S \simeq \mathcal{O}_S$  and  $H^1(S, \mathcal{O}_S) = 0$ . These surfaces are usually studied using their intersection lattice, that is,  $\text{Pic}(S) = H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$  with its associated intersection form. The classical Noether-Lefschetz theorem states that the Picard number, i.e., the rank of the Picard group, of a general quartic surface  $S$  in  $\mathbb{P}^3$  is 1. More generally it is known that the locus of surfaces with Picard number  $\rho$  has codimension  $\rho - 1$  in the Hilbert scheme of quartic surfaces.

Linear systems on K3 surfaces can be studied effectively thanks to the many vanishing theorems satisfied by nef divisors, i.e., divisors having non-negative intersection with every curve. For example, if  $D$  is a nef divisor with  $D^2 > 0$ , then the Kawamata-Vieweg vanishing theorem says that  $H^1(S, \mathcal{O}_S(D)) = H^2(S, \mathcal{O}_S(D)) = 0$ . In particular, this is satisfied when  $D$  is ample. If on the other hand  $D$  is nef and  $D^2 = 0$ ,  $D$  is base-point free and  $D \equiv kE$  for some smooth elliptic curve  $E$ . In this case  $h^1(S, \mathcal{O}_S(D)) = k - 1$  and  $H^2(S, \mathcal{O}_S(D)) = 0$  [32, Proposition 2.6]. In short, these results say that the dimensions of the cohomology groups  $H^i(S, \mathcal{O}_S(D))$  can be computed effectively using the Riemann-Roch formula  $\chi(\mathcal{O}_S(D)) = \frac{D^2}{2} + 2$ . To compute the cohomology groups for non-nef divisors, we will need the following lemma:

**Lemma 6.1.** *Let  $S$  be a smooth projective K3 surface with a smooth rational curve  $\Gamma$  and let  $D$  be a divisor such that  $D - \Gamma \neq 0$  is effective and  $d = -D \cdot \Gamma > 0$ .*

- *If  $H^1(S, \mathcal{O}_S(D - \Gamma)) \neq 0$ , then  $H^1(S, \mathcal{O}_S(D)) \neq 0$ .*
- *If  $d > 1$ , then  $H^1(S, \mathcal{O}_S(D)) \neq 0$ .*
- *If  $d = 1$  and  $H^1(S, \mathcal{O}_S(D - \Gamma)) = 0$ , then  $H^1(S, \mathcal{O}_S(D)) = 0$ .*

*Proof.* Taking cohomology of the exact sequence

$$0 \rightarrow \mathcal{O}_S(D - \Gamma) \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_\Gamma(-d) \rightarrow 0,$$

and using the fact that  $\Gamma \simeq \mathbb{P}^1$ , we get

$$h^1(\mathcal{O}_S(D)) = h^1(\mathcal{O}_S(D - \Gamma)) + h^1(\mathcal{O}_{\mathbb{P}^1}(-d)) = h^1(\mathcal{O}_S(D - \Gamma)) + d - 1$$

and the result follows. □

## 6.1 Quartic surfaces containing a line

The most natural way of constructing quartic surfaces with Picard number  $> 1$  is to impose that it should contain a curve  $\Gamma$  which is not the intersection of  $S$  with another surface (that is,  $\Gamma$  is not linearly equivalent to a multiple of the hyperplane section divisor). Here we consider the case where  $\Gamma$  is a line. Such quartic surfaces appeared in the work of Mori [28], who showed the following result: If there exists a smooth quartic surface  $S_0$  containing a nonsingular curve  $\Gamma_0$  of degree  $d$  and genus  $g$ , then there also exists a smooth quartic surface  $S$  containing a smooth curve  $\Gamma$  of the same degree and genus, such that  $\text{Pic}(S) = \mathbb{Z}\Gamma \oplus \mathbb{Z}H$ , where  $H$  is the hyperplane section. (See also [15, p. 138]).

**Proposition 6.2.** *There exists a smooth quartic K3 surface  $S \subset \mathbb{P}^3$  with  $\text{Pic}(S) = \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$  where  $\Gamma_1, \Gamma_2$  are smooth curves of genus 0 and 1 respectively, and intersection matrix given by*

$$\begin{pmatrix} \Gamma_1^2 & \Gamma_1 \cdot \Gamma_2 \\ \Gamma_1 \cdot \Gamma_2 & \Gamma_2^2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix}$$

Furthermore, the following hold:

- i) The hyperplane divisor is given by  $H \equiv \Gamma_1 + \Gamma_2$ .
- ii) For a smooth curve  $C$  with divisor class  $a\Gamma_1 + b\Gamma_2$ , we have

$$d = H \cdot C = a + 3b \quad g(C) = \frac{C^2}{2} + 1 = 3ab - a^2 + 1.$$

- iii) Any effective divisor class can be written as  $a\Gamma_1 + b\Gamma_2$  for non-negative integers  $a, b \geq 0$ .

- iv) A divisor class  $a\Gamma_1 + b\Gamma_2$  is nef if and only if  $3b \geq 2a \geq 0$ .

- v) If  $D \equiv a\Gamma_1 + b\Gamma_2$  is a divisor with  $3b \geq 2a > 0$  and  $a > 0$ , then the generic element in  $|D|$  is a smooth irreducible curve. Conversely, the classes of the irreducible divisors correspond to classes  $a\Gamma_1 + b\Gamma_2$  with  $3b \geq 2a > 0$  or  $(a, b) = (1, 0), (0, 1)$ .

*Proof.* Smooth quartic surfaces  $S_0$  containing a line  $\{x_0 = x_1 = 0\}$  are defined by a homogenous polynomial of the form  $F = x_0p + x_1q = 0$  where  $p, q \in k[x_0, x_1, x_2, x_3]$  are cubic forms. By Mori's result above there exists a smooth quartic surface  $S$  such that  $\text{Pic}(S)$  is generated by a smooth rational curve  $\Gamma_1$  and the hyperplane section  $H$ . By the adjunction formula, we have  $\Gamma_1^2 = -2$ . In fact the diophantine equation  $(xH + y\Gamma_1)^2 = 4x^2 + 2xy - 2y^2 = 2(2x - y)(x + y) = -2$  has the only solutions  $(0, \pm 1)$ , showing that  $\Gamma_1$  is the unique  $(-2)$ -curve on  $S$ . The class  $H - \Gamma_1$  has self-intersection 0 and is thus effective. Moreover, using Bertini's theorem and the adjunction formula, we see that the generic element  $\Gamma_2 \in |H - \Gamma_1|$  is a smooth elliptic curve. In the following, we fix smooth curves  $\Gamma_1, \Gamma_2$ , whose classes form a basis for  $\text{Pic}(S)$ .

ii) is elementary and follows from the adjunction formula. To prove iii) we claim that every effective divisor is linearly equivalent to a positive integral linear combination of  $\Gamma_1$  and  $\Gamma_2$ . Indeed, let  $D$  be any effective divisor class and write  $D = a\Gamma_1 + b\Gamma_2$  for integers  $a, b$ . We may assume that  $D \cdot \Gamma_1 \geq 0$  (otherwise  $\Gamma_1$  is a fixed component of the linear system  $|D|$  and we can instead consider  $D - \Gamma_1$ ). Then we have  $0 \leq D \cdot \Gamma_1 = 3b - 2a$  and  $0 \leq D \cdot \Gamma_2 = 3a$  implying that  $a, b \geq 0$ . Dually we have also shown that the nef cone is determined by the inequalities  $a \geq 0$  and  $3b \geq 2a$ , giving iii) and iv).

v): If  $C$  is an irreducible curve with  $C \neq \Gamma_1, \Gamma_2$ , then  $C$  is nef and  $C \cdot \Gamma_2 > 0$  (by the Hodge index theorem). So  $C \equiv a\Gamma_1 + b\Gamma_2$  with  $3b - 2a \geq 0$  and  $a > 0$ . Conversely, if these conditions are satisfied, the divisor  $D = a\Gamma_1 + b\Gamma_2$  is base-point free [32, Corollary 3.2] and hence by Bertini's theorem the general element in  $|D|$  is smooth and irreducible. This shows that the classes listed in Theorem 1.1 actually represent smooth, irreducible curves.  $\square$

For the existence of such a K3 surface one could also employ a result of Nikulin [31] which states that for any even lattice of signature  $(1, \rho - 1)$  with  $\rho \leq 10$ , there exists a smooth projective K3 surface with this intersection form. Using this, and the embedding criteria of Saint-Donant [32], one can show that any surface with intersection matrix as above embeds as a smooth quartic surface.

In Remark 3.2 it is necessary to find effective divisors  $D = a\Gamma_1 + b\Gamma_2$  such that  $H^1(S, \mathcal{O}_S(D)) \neq 0$ . If  $D$  is effective, then by the proposition above, we must have  $a, b \geq 0$ . If  $a = 0$  then  $h^1(S, \mathcal{O}_S(D)) = h^1(S, \mathcal{O}_S(b\Gamma_2)) = \min\{b - 1, 0\}$ . We will assume  $a > 0$ . If now  $d = -D \cdot \Gamma_1 = 2a - 3b \leq 0$ , then  $D$  is nef and  $D^2 = a(3b - 2a) + 3ab > 0$  and so  $h^1(S, \mathcal{O}_S(D)) = 0$ . If  $d = 1$ , then  $a > 1$  (due to  $1 = 2a - 3b$ ) and  $D - \Gamma_1$  is nef with  $(D - \Gamma_1)^2 > 0$  and so  $H^1(\mathcal{O}_S(D - \Gamma_1)) = 0$  and consequently  $H^1(S, \mathcal{O}_S(D)) = 0$  by Lemma 6.1. If  $d > 1$  then we have  $H^1(S, \mathcal{O}_S(D)) \neq 0$ . Hence we obtain

**Proposition 6.3.** *Let  $S$  be a quartic surface with  $\text{Pic}(S) = \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$  and  $\Gamma_1, \Gamma_2$  as above. Suppose  $D = a\Gamma_1 + b\Gamma_2$  is an effective divisor class with  $a > 0$ . Then*

$$H^1(S, \mathcal{O}_S(D)) \neq 0 \quad \text{if and only if} \quad 2a > 3b + 1. \quad (19)$$

Moreover, if  $a = 0$ , then  $h^1(S, \mathcal{O}_S(D)) = \max\{b - 1, 0\}$ .

## 6.2 Other surfaces

The surface appearing in Theorem 1.1 is only one example of a quartic surface for which we can use the theory of this paper to describe the smoothness properties of the components of the Hilbert scheme. In fact, by the result of Mori quoted above, it is clear that there should exist many such examples, but finding ones with irreducible curves satisfying the bound (1) seems more difficult. Nevertheless, let us finish by giving the main details for the smooth quartic surface appearing in Proposition 4.3.

Consider a homogenous quartic form of the form  $F = x_0p + q_1q_2$  where  $q_1, q_2$  are quadrics defining the plane conics and  $p$  is a cubic.  $F$  defines a smooth quartic surface  $S_0 \subset \mathbb{P}^3$ , where the hyperplane section splits into two plane conics ( $\{x_0 = q_1 = 0\}$  and  $\{x_0 = q_2 = 0\}$ ). Then by Mori's result above one obtains a smooth quartic surface with the intersection matrix  $(\Gamma_i \cdot \Gamma_j) = \begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$ . (Again we choose the basis  $\{\Gamma_1, \Gamma_2\}$  for  $\text{Pic}(S)$  rather than  $\{H, \Gamma_1\}$ .) As before, one can show that  $\Gamma_1, \Gamma_2$  define smooth irreducible  $(-2)$ -curves which generate the semigroup of effective divisors.

We also get the following result:

**Proposition 6.4.** *There exists a smooth quartic surface  $S$  with  $\text{Pic}(S) = \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$  and  $\Gamma_1, \Gamma_2$  as above.*

i) *The hyperplane divisor is given by  $H \equiv \Gamma_1 + \Gamma_2$ .*

ii) *For a smooth curve  $C$  with divisor class  $a\Gamma_1 + b\Gamma_2$ , we have*

$$d = H \cdot C = 2a + 2b, \quad g(C) = 4ab - a^2 - b^2 + 1$$

iii) *Any effective divisor class can be written as  $a\Gamma_1 + b\Gamma_2$  for non-negative integers  $a, b \geq 0$ .*

iv) *A divisor class  $a\Gamma_1 + b\Gamma_2$  is nef if and only if  $\frac{b}{2} \leq a \leq 2b$ .*

v) *If  $D \equiv a\Gamma_1 + b\Gamma_2$  is a divisor with  $a, b > 0$ , then  $H^1(S, \mathcal{O}_S(D)) = 0$  if and only if  $D$  is nef.*

vi) *If  $D \equiv a\Gamma_1 + b\Gamma_2$  is a divisor with  $0 < \frac{b}{2} \leq a \leq 2b$ , then the generic element in  $|D|$  is a smooth irreducible curve. Conversely, the classes of the irreducible curves correspond to classes  $a\Gamma_1 + b\Gamma_2$  satisfying  $a, b > 0$  and  $\frac{b}{2} \leq a \leq 2b$  or  $(a, b) = (1, 0), (0, 1)$ .*



*Proof.* The first part of the theorem and parts *i*) – *iii*) follow as in the proof of Proposition 6.2. If  $D = a\Gamma_1 + b\Gamma_2$  is nef and non-zero, then intersecting with  $\Gamma_1$  and  $\Gamma_2$  gives the above inequality for *iv*). If the condition in *v*) is satisfied, we have  $D^2 = 8ab - 2a^2 - 2b^2 \geq 4b^2 - \frac{1}{2}b^2 - 2b^2 > 0$  and so by the Kawamata-Viehweg vanishing theorem,  $H^1(S, \mathcal{O}_S(D)) = 0$ . Conversely, if  $D$  is not nef, then we can without loss of generality assume  $d = -D \cdot \Gamma_1 > 0$ . But  $d$  must be an even number, hence  $d > 1$  and so  $H^1(S, \mathcal{O}_S(D)) \neq 0$  by Lemma 6.1.  $\square$

Note that Proposition *v*) above allows us to see exactly when  $h^1(\mathcal{I}_C(4)) = h^1(\mathcal{O}_S(4H - C)) = h^1(\mathcal{O}_S(C - 4H)) = 0$ , as needed in Remark 4.4.

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